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Generic gauge fields in the Hubbard model: emergence of pairing interaction

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The spin-rotationally invariant SU(2) approach to the Hubbard model is extended to accommodate the charge degrees of freedom. Both U(1) and SU(2) gauge transformation are useed to factorize the charge and spin contribution to the original electron operator in terms of the emergent gauge fields. It is shown that these fields play a similar role as phonons in the BCS theory: they provide the "glue" for fermion pairing. By tracing out gauge bosons the form of paired states is established and the role of antiferromagnetic correlations is explicated.

To understand the physics of strongly correlated (SC) systems one frequently employs a slave particle (SP) decomposition of the electron operator.[1,2] Analytic theory behind this formulation hinges on the treatment of the constraint of no double occupation based on the assumption that the on-site interaction energy U can be renormalized to infinity. The redundancy in representations used to enforce the constraint naturally leads to various gauge theories.[3] It is sometimes supposed that slave particles can be liberated at low energies, in which case the slave-boson and fermion degrees of freedom take on a physical meaning with the spin-charge separation as a result. However, it was shown that gauge theories associated with SP representations of correlated electrons, such as the $t-J$ model, are always confining.[4] The reason is that the slave-particle gauge theory is infinitely strongly coupled– there is no intrinsic kinetic energy for the gauge field. Therefore, the slave particle representations of correlated electron models are not belonging to the class of *generic* gauge theories. Although matter fields which couple to the non-generic SP gauge field might generate gauge dynamics in the low-energy effective action,[5] the problem remains: how to *derive* emergent gauge fields from the microscopic formulation of the theory of SC electrons and not just to write down a Lagrangian that contains them - as in the SP approach. This issue is of basic concern because the understanding of the mechanism of superconductivity in cuprates requires the knowledge of bosons mediating the pairing as well as the nature of the paired states. Here, the underlying attraction force appears very puzzling since it is hard to reconcile the microscopic attractive interaction with the completely repulsive bare electron-electron forces.

In the present paper we extend the SU(2) spin-rotationally invariant approach to the Hubbard model,[6] which makes no assumptions regarding the magnitude of the Coulomb energy U, to accommodate on equal footing the *charge* degrees of freedom. Using U(1) and SU(2) transformation we explicitly factorize the charge and spin contribution to the original electron operator in terms of the corresponding *emergent* gauge fields. We show that these fields play a similar role as phonons in the BCS theory: they provide the "glue" for fermion pairing in the SC system. By tracing out gauge bosons we explicitly calculate and the form of paired states and explicate the role of antiferromagnetic (AF) correlations.

Our starting point is the purely fermionic Hubbard Hamiltonian $\mathcal{H} \equiv \mathcal{H}_t + \mathcal{H}_U$:

$$\mathcal{H} = -t \sum_{\langle \mathbf{r}\mathbf{r}'\rangle,\alpha} [c^\dagger_\alpha(\mathbf{r}) c_\alpha(\mathbf{r}') + \text{h.c.}] + \sum_{\mathbf{r}} U n_\uparrow(\mathbf{r}) n_\downarrow(\mathbf{r}). \quad (1)$$

Here, $\langle \mathbf{r}, \mathbf{r}'\rangle$ runs over the nearest-neighbor (n.n.) sites, t is the hopping amplitude, U stands for the Coulomb repulsion, while the operator $c^\dagger_\alpha(\mathbf{r})$ creates an electron with spin $\alpha =\uparrow,\downarrow$ at the lattice site $\mathbf{r}$. Furthermore, $n(\mathbf{r}) = n_\uparrow(\mathbf{r}) + n_\downarrow(\mathbf{r})$ is the number operator, where $n_\alpha(\mathbf{r}) = c^\dagger_\alpha(\mathbf{r}) c_\alpha(\mathbf{r})$. Usually, working in the grand canonical ensemble a term $-\mu \sum_{\mathbf{r}} n(\mathbf{r})$ is added to $\mathcal{H}$ in Eq.(1) with μ being the chemical potential . It is customary to introduce Grassmann fields, $c_\alpha(\mathbf{r}\tau)$ depending on the "imaginary time" $0 \le \tau \le \beta \equiv 1/k_B T$, (with T being the temperature) that satisfy the anti–periodic condition $c_\alpha(\mathbf{r}\tau) = -c_\alpha(\mathbf{r}\tau + \beta)$, to write the path integral for the statistical sum $\mathcal{Z} = \int [\mathcal{D}\bar{c}\mathcal{D}c]\, e^{-\mathcal{S}[\bar{c},c]}$ with the fermionic action

$$\mathcal{S}[\bar{c}, c] = \mathcal{S}_B[\bar{c}, c] + \int_0^\beta d\tau \mathcal{H}[\bar{c}, c], \quad (2)$$

that contains the fermionic Berry term

$$\mathcal{S}_B[\bar{c}, c] = \sum_{\mathbf{r}\alpha} \int_0^\beta d\tau \bar{c}_\alpha(\mathbf{r}\tau) \partial_\tau c_\alpha(\mathbf{r}\tau). \quad (3)$$

For the SC system it is crucial to construct a covariant formulation of the theory which naturally preserves the spin-rotational symmetry present in the Hubbard Hamiltonian. For this purpose the density–density product in Eq.(1) we write, following Ref.6, in a spin-rotational invariant way:

$$\mathcal{H}_U = U \sum_{\mathbf{r}} \left\{ \frac{1}{4} n^2(\mathbf{r}\tau) - [\mathbf{\Omega}(\mathbf{r}\tau) \cdot \mathbf{S}(\mathbf{r}\tau)]^2 \right\}, \quad (4)$$

where $S^a(\mathbf{r}\tau) = \frac{1}{2} \sum_{\alpha\alpha'} c^\dagger_\alpha(\mathbf{r}\tau) \hat{\sigma}^a_{\alpha\alpha'} c_{\alpha'}(\mathbf{r}\tau)$ denotes the vector spin operator ($a = x, y, z$) with $\hat{\sigma}^a$ being the Pauli matrices. The unit vector $\mathbf{\Omega}(\mathbf{r}\tau) =$

$[\sin\vartheta(\mathbf{r}\tau)\cos\varphi(\mathbf{r}\tau), \sin\vartheta(\mathbf{r}\tau)\sin\varphi(\mathbf{r}\tau), \cos\vartheta(\mathbf{r}\tau)]$ written in terms of polar angles labels varying in space-time spin quantization axis. The spin–rotation invariance is made explicit by performing the angular integration over $\mathbf{\Omega}(\mathbf{r}\tau)$ at each site and time. By decoupling spin and charge density terms in Eq.(4) using auxiliary fields $\varrho(\mathbf{r}\tau)$ and $iV(\mathbf{r}\tau)$ respectively, we write down the partition function in the form

$$\begin{aligned} \mathcal{Z} &= \int [\mathcal{D}\mathbf{\Omega}] \int [\mathcal{D}V\mathcal{D}\varrho] \int [\mathcal{D}\bar{c}\mathcal{D}c] \times \\ &\times \; e^{-\mathcal{S}[\mathbf{\Omega},V,\varrho,\bar{c},c]}. \end{aligned} \tag{5}$$

where $[\mathcal{D}\mathbf{\Omega}] \equiv \prod_{\mathbf{r}\tau_k} \frac{\sin\vartheta(\mathbf{r}\tau_k)d\vartheta(\mathbf{r}\tau_k)d\varphi(\mathbf{r}\tau_k)}{4\pi}$ is the spin-angular integration measure. The effective action reads:

$$\begin{aligned} \mathcal{S}\left[\mathbf{\Omega}, V, \varrho, \bar{c}, c\right] &= \sum_{\mathbf{r}} \int_0^\beta d\tau \left[\frac{\varrho^2(\mathbf{r}\tau)}{U} + \frac{V^2(\mathbf{r}\tau)}{U} \right. \\ &+ \left. iV(\mathbf{r}\tau)n(\mathbf{r}\tau) + 2\varrho(\mathbf{r}\tau)\mathbf{\Omega}(\mathbf{r}\tau)\cdot\mathbf{S}(\mathbf{r}\tau)\right] \\ &+ \mathcal{S}_B[\bar{c},c] + \int_0^\beta d\tau \mathcal{H}_t[\bar{c},c]. \end{aligned} \tag{6}$$

Simple Hartree-Fock (HF) theory will not work for a Hubbard model in which U is the largest energy in the problem. One has to isolate strongly fluctuating modes generated by the Hubbard term according to the charge U(1) and spin SU(2) symmetries. To this end we write the fluctuating "imaginary chemical potential" $iV(\mathbf{r}\tau)$ as a sum of a static $V_0(\mathbf{r})$ and periodic function $V(\mathbf{r}\tau) = V_0(\mathbf{r}) + \tilde{V}(\mathbf{r}\tau)$ using Fourier series

$$\tilde{V}(\mathbf{r}\tau) = \frac{1}{\beta}\sum_{n=1}^{\infty}[\tilde{V}(\mathbf{r}\omega_n)e^{i\omega_n\tau} + c.c.] \tag{7}$$

with $\omega_n = 2\pi n/\beta$ ($n = 0, \pm1, \pm2$) being the (Bose) Matsubara frequencies. Now, we introduce the U(1) *phase* field $\phi(\mathbf{r}\tau)$ via the Faraday–type relation

$$\dot{\phi}(\mathbf{r}\tau) \equiv \frac{\partial\phi(\mathbf{r}\tau)}{\partial\tau} = \tilde{V}(\mathbf{r}\tau). \tag{8}$$

Since the homotopy group $\pi_1[U(1)]$ forms a set of integers, discrete configurations of $\phi(\mathbf{r}\tau)$ matter, for which $\phi(\mathbf{r}\beta) - \phi(\mathbf{r}0) = 2\pi m(\mathbf{r})$, where $m(\mathbf{r}) = 0, \pm1, \pm2, \ldots$ Thus the decomposition of the charge field $V(\mathbf{r}\tau)$ conforms with the basic $m = 0$ topological sector since $\int_0^\beta \dot{\phi}(\mathbf{r}\tau) = \int_0^\beta \tilde{V}(\mathbf{r}\tau) \equiv 0$. Furthermore, by performing the local gauge transformation to the *new* fermionic variables $f_\alpha(\mathbf{r}\tau)$:

$$\begin{bmatrix} c_\alpha(\mathbf{r}\tau) \\ \bar{c}_\alpha(\mathbf{r}\tau) \end{bmatrix} = \begin{bmatrix} z(\mathbf{r}\tau) & 0 \\ 0 & \bar{z}(\mathbf{r}\tau) \end{bmatrix} \begin{bmatrix} f_\alpha(\mathbf{r}\tau) \\ \bar{f}_\alpha(\mathbf{r}\tau) \end{bmatrix} \tag{9}$$

where the unimodular parameter $|z(\mathbf{r}\tau)|^2 = 1$ satisfies $z(\mathbf{r}\tau) = e^{i\phi(\mathbf{r}\tau)}$, we remove the imaginary term $i\int_0^\beta d\tau\tilde{V}(\mathbf{r}\tau)n(\mathbf{r}\tau)$ for all the Fourier modes of the $V(\mathbf{r}\tau)$ field, except for the zero frequency. Subsequent SU(2) transformation from $f_\alpha(\mathbf{r}\tau)$ to $h_\alpha(\mathbf{r}\tau)$ operators,

$$\begin{bmatrix} f_1(\mathbf{r}\tau) \\ f_2(\mathbf{r}\tau) \end{bmatrix} = \begin{bmatrix} \zeta_1(\mathbf{r}\tau) & -\bar{\zeta}_2(\mathbf{r}\tau) \\ \zeta_2(\mathbf{r}\tau) & \bar{\zeta}_1(\mathbf{r}\tau) \end{bmatrix} \begin{bmatrix} h_1(\mathbf{r}\tau) \\ h_2(\mathbf{r}\tau) \end{bmatrix} \tag{10}$$

with the constraint $|\zeta_1(\mathbf{r}\tau)|^2 + |\zeta_2(\mathbf{r}\tau)|^2 = 1$ takes away the rotational dependence on $\mathbf{\Omega}(\mathbf{r}\tau)$ in the spin sector. This is done by means of the Hopf map $\mathbf{R}(\mathbf{r}\tau)\hat{\sigma}^z\mathbf{R}^\dagger(\mathbf{r}\tau) = \hat{\boldsymbol{\sigma}}\cdot\mathbf{\Omega}(\mathbf{r}\tau)$ that is based on the enlargement from two-sphere S_2 to the three-sphere $S_3 \sim$ SU(2). The unimodular constraint can be resolved by using the parametrization

$$\begin{aligned} \zeta_1(\mathbf{r}\tau) &= e^{-\frac{i}{2}[\varphi(\mathbf{r}\tau)+\chi(\mathbf{r}\tau)]}\cos\left[\frac{\vartheta(\mathbf{r}\tau)}{2}\right] \\ \zeta_2(\mathbf{r}\tau) &= e^{\frac{i}{2}[\varphi(\mathbf{r}\tau)-\chi(\mathbf{r}\tau)]}\sin\left[\frac{\vartheta(\mathbf{r}\tau)}{2}\right] \end{aligned} \tag{11}$$

with the Euler angular variables $\varphi(\mathbf{r}\tau), \vartheta(\mathbf{r}\tau)$ and $\chi(\mathbf{r}\tau)$, respectively. Here, the extra variable $\chi(\mathbf{r}\tau)$ represents the U(1) gauge freedom of the theory as a consequence of $S_2 \to S_3$ mapping. One can summarize Eqs (9) and (10) by the single joint gauge transformation exhibiting electron operator factorization

$$c_\alpha(\mathbf{r}\tau) = \sum_{\alpha'} z(\mathbf{r}\tau)R_{\alpha\alpha'}(\mathbf{r}\tau)h_{\alpha'}(\mathbf{r}\tau), \tag{12}$$

where $\mathbf{R}(\mathbf{r}\tau) = e^{-i\hat{\sigma}_z\varphi(\mathbf{r}\tau)/2}e^{-i\hat{\sigma}_y\vartheta(\mathbf{r}\tau)/2}e^{-i\hat{\sigma}_z\chi(\mathbf{r}\tau)/2}$ is a unitary matrix which rotates the spin-quantization axis at site $\mathbf{r}$ and time τ. Eq.(12) reflects the composite nature of the interacting electron formed from bosonic spinorial and charge degrees of freedom given by $R_{\alpha\alpha'}(\mathbf{r}\tau)$ and $z(\mathbf{r}\tau)$, respectively as well as remaining fermionic part $h_\alpha(\mathbf{r}\tau)$. In the new variables the action in Eq.(6) assumes the form

$$\begin{aligned} \mathcal{S}\left[\mathbf{\Omega}, \phi, \varrho, \bar{h}, h\right] &= \mathcal{S}_B[\bar{h},h] + \int_0^\beta d\tau\mathcal{H}_{\mathbf{\Omega},\phi}[\rho,\bar{h},h] \\ &+\mathcal{S}_0\left[\phi\right] + 2\sum_{\mathbf{r}}\int_0^\beta d\tau\mathbf{A}(\mathbf{r}\tau)\cdot\mathbf{S}_h(\mathbf{r}\tau), \end{aligned} \tag{13}$$

where $\mathbf{S}_h(\mathbf{r}\tau) = \frac{1}{2}\sum_{\alpha\gamma}\bar{h}_\alpha(\mathbf{r}\tau)\hat{\boldsymbol{\sigma}}_{\alpha\gamma}h_\gamma(\mathbf{r}\tau)$. Furthermore,

$$S_0[\phi] = \sum_{\mathbf{r}}\int_0^\beta d\tau\left[\frac{\dot{\phi}^2(\mathbf{r}\tau)}{U} + \frac{2\mu}{iU}\dot{\phi}(\mathbf{r}\tau)\right] \tag{14}$$

stands for the kinetic and Berry term of the U(1) phase field in the charge sector. The SU(2) gauge transformation in Eq.(10) and the fermionic Berry term in Eq.(3) generate SU(2) potentials given by $\mathbf{R}^\dagger(\mathbf{r}\tau)\partial_\tau\mathbf{R}(\mathbf{r}\tau) =$

$-\hat{\boldsymbol{\sigma}}\cdot\mathbf{A}(\mathbf{r}\tau)$, where

$$\begin{aligned} A^x(\mathbf{r}\tau) &= \frac{i}{2}\dot{\vartheta}(\mathbf{r}\tau)\sin\chi(\mathbf{r}\tau) \\ &- \frac{i}{2}\dot{\varphi}(\mathbf{r}\tau)\sin\theta(\mathbf{r}\tau)\cos\chi(\mathbf{r}\tau) \\ A^y(\mathbf{r}\tau) &= \frac{i}{2}\dot{\vartheta}(\mathbf{r}\tau)\cos\chi(\mathbf{r}\tau) \\ &+ \frac{i}{2}\dot{\varphi}(\mathbf{r}\tau)\sin\theta(\mathbf{r}\tau)\sin\chi(\mathbf{r}\tau) \\ A^z(\mathbf{r}\tau) &= \frac{i}{2}\dot{\varphi}(\mathbf{r}\tau)\cos\vartheta(\mathbf{r}\tau)+\frac{i}{2}\dot{\chi}(\mathbf{r}\tau). \end{aligned} \tag{15}$$

In analogy to the charge U(1) field the SU(2) spin system exhibits emergent dynamics. By integrating out fermions the last term in Eq.(13) will generate the kinetic term for the SU(2) rotors $\mathcal{S}_0[\mathbf{\Omega}]=-(1/\mathcal{E}_s)\sum_{\mathbf{r}}\int_0^\beta d\tau\mathbf{A}(\mathbf{r}\tau)\cdot\mathbf{A}(\mathbf{r}\tau)$ in the form

$$\begin{aligned} \mathcal{S}_0[\mathbf{\Omega}] = \frac{1}{4\mathcal{E}_s}\sum_{\mathbf{r}}\int_0^\beta d\tau\left[\dot{\vartheta}^2(\mathbf{r}\tau)+\dot{\varphi}^2(\mathbf{r}\tau)\right. \\ \left.+\dot{\chi}^2(\mathbf{r}\tau)+2\dot{\varphi}(\mathbf{r}\tau)\dot{\chi}(\mathbf{r}\tau)\cos\vartheta(\mathbf{r}\tau)\right] \end{aligned} \tag{16}$$

with $\mathcal{E}_s$ being of order of U close to half filling. The first order term in $\mathbf{A}(\mathbf{r}\tau)$ fields gives rise to the usual spin Berry contribution. If we work in Dirac "north pole" gauge $\chi(\mathbf{r}\tau)=-\varphi(\mathbf{r}\tau)$ one recoverss the familiar form $\mathcal{S}_B[\mathbf{\Omega}]=-i(\rho/U)\sum_{\mathbf{r}}\int_0^\beta d\tau\dot{\varphi}(\mathbf{r}\tau)[1-\cos\vartheta(\mathbf{r}\tau)]$. The fermionic sector, in turn, is governed by the effective Hamiltonian

$$\begin{aligned} \mathcal{H}_{\mathbf{\Omega},\phi} &= \sum_{\mathbf{r}}\varrho(\mathbf{r}\tau)[\bar{h}_\uparrow(\mathbf{r}\tau)h_\uparrow(\mathbf{r}\tau)-\bar{h}_\downarrow(\mathbf{r}\tau)h_\downarrow(\mathbf{r}\tau)] \\ &-t\sum_{\substack{\langle\mathbf{r},\mathbf{r}'\rangle\\ \alpha\gamma}}\bar{z}(\mathbf{r}\tau)z(\mathbf{r}'\tau)\left[\mathbf{R}^\dagger(\mathbf{r}\tau)\mathbf{R}(\mathbf{r}'\tau)\right]_{\alpha\gamma}\bar{h}_\alpha(\mathbf{r}\tau)h_\gamma(\mathbf{r}'\tau) \\ &-\bar{\mu}\sum_{\mathbf{r}\alpha}\bar{h}_\alpha(\mathbf{r}\tau)h_\alpha(\mathbf{r}\tau), \end{aligned} \tag{17}$$

where $\bar{\mu}=\mu-n_fU/2$ is the chemical potential with a Hartree shift originating from the saddle-point value of the static variable $V_0(\mathbf{r})$ with $n_h=n_{h\uparrow}+n_{h\downarrow}$ and $n_{h\alpha}=\langle\bar{h}_\alpha(\mathbf{r}\tau)h_\alpha(\mathbf{r}\tau)\rangle$. The chief merit of the gauge transformation in Eq.(12) is that we have managed to cast the SC problem into a system of non-interacting h fermions submerged in the bath of strongly fluctuating U(1) and SU(2) gauge potentials coupled to fermions via hopping term plus Zeeman-type contribution with the massive field $\varrho(\mathbf{r}\tau)$. In the AF phase, at the half-filling, it assumes the staggered form $\varrho(\mathbf{r}\tau)=\Delta_c(-1)^{\mathbf{r}}$ with Δ_c being the charge gap $\Delta_c\sim U/2$ for $U/t\gg 1$. However, a nonzero value of Δ_c does not imply the existence of AF long–range order. For this the angular degrees of freedom $\mathbf{\Omega}(\mathbf{r}\tau)$ have also to be ordered, whose low-lying excitations are in the form of spin waves.

It is well known that phonons play the role of the "glue" that is responsible for the formation of Cooper pairs in conventional superconductors. Now we show that U(1) *and* SU(2) emergent gauge fields, the collective high energy modes in the SC system, take over the task which was carried out by phonons in BCS superconductors. In a way similar to phonons these gauge fields couple to the fermion density type term via the amplitude t, see Eq.(17). Now we evaluate the effective interaction between fermions by tracing out the gauge degrees of freedom. To this end we write the partition function as $\mathcal{Z}=\int[\mathcal{D}\bar{h}\mathcal{D}h]e^{-\mathcal{S}[\bar{h},h]}$, where

$$\mathcal{S}[\bar{h},h] \;=\; -\ln\int[\mathcal{D}\mathbf{\Omega}\mathcal{D}\phi]\,e^{-\mathcal{S}[\mathbf{\Omega},\phi,\bar{h},h]} \tag{18}$$

generates cumulant expansion for the effective fermionic action. We concentrate on the second order term in the hoping amplitude t containing four fermion operators:

$$\begin{aligned} \mathcal{S}^{(2)}[\bar{h},h] = \; -\frac{t^2}{2}\int_0^\beta d\tau d\tau'\left\langle\sum_{|\mathbf{r}_1-\mathbf{r}_1'|=n.n.}\bar{z}(\mathbf{r}_1\tau)z(\mathbf{r}_1'\tau)\sum_{\alpha\alpha'}\left[\mathbf{R}^\dagger(\mathbf{r}_1\tau)\mathbf{R}(\mathbf{r}'_1\tau)\right]_{\alpha\alpha'}\bar{h}_\alpha(\mathbf{r}_1\tau)h_{\alpha'}(\mathbf{r}_1'\tau)\times\right. \\ \left.\times\sum_{|\mathbf{r}_2-\mathbf{r}_2'|=n.n.}\bar{z}(\mathbf{r}_2\tau')z(\mathbf{r}_2'\tau')\sum_{\gamma\gamma'}\left[\mathbf{R}^\dagger(\mathbf{r}_2\tau')\mathbf{R}(\mathbf{r}_2'\tau')\right]_{\gamma\gamma'}\bar{h}_\gamma(\mathbf{r}_2\tau')h_{\gamma'}(\mathbf{r}_2'\tau')\right\rangle, \end{aligned} \tag{19}$$

where $\langle\dots\rangle$ denotes averaging over U(1) and SU(2) gauge fields. The averaging in the charge sector is performed with the use of the U(1) phase action in Eq.(14) to give

$$\begin{aligned} &\langle\bar{z}(\mathbf{r}_1\tau)z(\mathbf{r}_1'\tau)\bar{z}(\mathbf{r}_2\tau')z(\mathbf{r}_2'\tau')\rangle \\ &\simeq(\delta_{\mathbf{r_1},\mathbf{r}_1'}\delta_{\mathbf{r_2},\mathbf{r}_2'}+\delta_{\mathbf{r_1},\mathbf{r}_2'}\delta_{\mathbf{r}_1',\mathbf{r_2}})\times \\ &\times\exp\left\{-\frac{U}{2}\left[|\tau-\tau'|-\frac{(\tau-\tau')^2}{\beta}\right]\right\}. \end{aligned} \tag{20}$$

Equation (20) reflects the local (in space) nature of

charge excitation and contains only the non-topological part of the four-point charge correlator. Away from half-filling the dynamics of spin variables is slower as compared to the charge counterparts, allowing to treat SU(2) variables as local in time $\mathbf{R}(\mathbf{r}\tau') = \mathbf{R}(\mathbf{r}\tau) + (\tau' - \tau)\partial_\tau \mathbf{R}(\mathbf{r}\tau) + O[(\tau'-\tau)^2]$. Furthermore, in the low temperature limit (on the energy scale given by U), by making use of the formula

$$\lim_{\tau\to 0}\int_0^\beta d\tau' e^{-\frac{|\tau-\tau'|U}{2}} = \frac{2}{U} - \frac{2e^{-\frac{\beta U}{2}}}{U} \tag{21}$$

we arrive at

$$\mathcal{S}^{(2)}[\bar{h},h] = -\frac{t^2}{U}\int_0^\beta d\tau \sum_{\langle \mathbf{r},\mathbf{r}'\rangle}\sum_{\substack{\alpha\alpha'\\ \gamma\gamma'}} [\bar{h}_\alpha(\mathbf{r}'\tau)h_{\alpha'}(\mathbf{r}\tau)]^\dagger \times \langle M_{\alpha'\alpha;\gamma\gamma'}(\mathbf{r}\tau;\mathbf{r}'\tau|\mathbf{r}'\tau\mathbf{r}\tau)\rangle\, \bar{h}_\gamma(\mathbf{r}'\tau)h_{\gamma'}(\mathbf{r}\tau), \tag{22}$$

where $\langle \dots \rangle$ denotes averaging over the remaining spin-angular variables and

$$M_{\alpha'\alpha;\gamma\gamma'} = \left[\mathbf{R}^\dagger(\mathbf{r}\tau)\mathbf{R}(\mathbf{r}'\tau)\right]_{\alpha'\alpha}\left[\mathbf{R}^\dagger(\mathbf{r}'\tau)\mathbf{R}(\mathbf{r}\tau)\right]_{\gamma\gamma'}. \tag{23}$$

Now, employing the composition formula for rotational matrices[7]

$$\mathbf{R}^\dagger(\mathbf{r}\tau)\mathbf{R}(\mathbf{r}'\tau) = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{\frac{i}{2}\Phi}\Upsilon_+ & e^{\frac{i}{2}\bar{\Phi}}\Upsilon_- \\ -e^{-\frac{i}{2}\bar{\Phi}}\Upsilon_- & e^{-\frac{i}{2}\Phi}\Upsilon_+ \end{bmatrix} \tag{24}$$

with $\Upsilon_\pm(\mathbf{r}\tau,\mathbf{r}'\tau) = \sqrt{1 \pm \mathbf{\Omega}(\mathbf{r}\tau)\cdot\mathbf{\Omega}(\mathbf{r}'\tau)}$, where $\Phi \equiv \Phi[\mathbf{\Omega}(\mathbf{r}\tau),\mathbf{\Omega}(\mathbf{r}'\tau),\mathbf{z}]$ is the signed solid angle spanned by the vectors $\mathbf{\Omega}(\mathbf{r}\tau),\mathbf{\Omega}(\mathbf{r}'\tau)$ and $\mathbf{z}$ with $\bar{\Phi} = \Phi[\mathbf{\Omega}(\mathbf{r}\tau),-\mathbf{\Omega}(\mathbf{r}'\tau)] - 2\varphi(\mathbf{r}\tau)$, we finally conclude that

$$\mathcal{S}^{(2)}[\bar{h},h] = -\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\int_0^\beta d\tau \left[J_A\bar{\mathcal{A}}(\mathbf{r}\tau\mathbf{r}'\tau)\mathcal{A}(\mathbf{r}\tau\mathbf{r}'\tau) + J_F\bar{\mathcal{F}}(\mathbf{r}\tau\mathbf{r}'\tau)\mathcal{F}(\mathbf{r}\tau\mathbf{r}'\tau)\right], \tag{25}$$

where

$$J_{A/F} = \frac{2t^2}{U}\left[1 \mp \langle \mathbf{\Omega}(\mathbf{r}\tau)\cdot\mathbf{\Omega}(\mathbf{r}'\tau)\rangle\right] \tag{26}$$

and

$$\mathcal{A}(\mathbf{r}\tau\mathbf{r}'\tau) = \frac{h_\uparrow(\mathbf{r}\tau)h_\downarrow(\mathbf{r}'\tau) - h_\downarrow(\mathbf{r}\tau)h_\uparrow(\mathbf{r}'\tau)}{\sqrt{2}}$$
$$\mathcal{F}(\mathbf{r}\tau\mathbf{r}'\tau) = \frac{\bar{h}_\uparrow(\mathbf{r}\tau)h_\uparrow(\mathbf{r}'\tau) + \bar{h}_\downarrow(\mathbf{r}\tau)h_\downarrow(\mathbf{r}'\tau)}{\sqrt{2}} \tag{27}$$

being the valence bond operators.[8] The rotational invariance of the right-hand side in Eq.(25) is manifest since $-\bar{\mathcal{A}}(\mathbf{r}\tau\mathbf{r}'\tau)\mathcal{A}(\mathbf{r}\tau\mathbf{r}'\tau) = \mathbf{S}_h(\mathbf{r}\tau)\cdot\mathbf{S}_h(\mathbf{r}'\tau) - \frac{1}{4}$ and $\bar{\mathcal{F}}(\mathbf{r}\tau\mathbf{r}'\tau)\mathcal{F}(\mathbf{r}\tau\mathbf{r}'\tau) = \mathbf{S}_h(\mathbf{r}\tau)\cdot\mathbf{S}_h(\mathbf{r}'\tau) + \frac{1}{4}$, respectively. The effective non-retarded interaction $J_A > 0$ in front of the $\bar{\mathcal{A}}(\mathbf{r}\tau\mathbf{r}'\tau)\mathcal{A}(\mathbf{r}\tau\mathbf{r}'\tau)$ term constitutes the attractive potential for fermion pairing. By HF decoupling of the four-fermion term in Eq.(25) one obtains the "d-wave" solution for the singlet pairing gap $\Delta_d(\mathbf{r}\mathbf{r}') = \langle\bar{\mathcal{A}}(\mathbf{r}\tau\mathbf{r}'\tau)\rangle$. The role of antiferromagnetic correlations is also apparent due to the presence of spin-angular correlation function in Eq.(26). For example, in a fully developed AF background $\mathbf{\Omega}(\mathbf{r}\tau)\cdot\mathbf{\Omega}(\mathbf{r}'\tau) = -1$ for $\mathbf{r}$ and $\mathbf{r}'$ on neighboring sites, so that $J_A = 4t^2/U$ and $J_F = 0$, thus promoting "d-wave" pairing. The expectation value of the second valence bond operator $\langle\bar{\mathcal{F}}(\mathbf{r}\tau\mathbf{r}'\tau)\rangle$ competes with $\Delta_d(\mathbf{r}\mathbf{r}')$, since it enhances the kinetic energy of fermions, which eventually results in suppression of the d-wave gap at higher doping level. The actual strength of the effective interaction in Eq.(26) requires that the quantity $\langle\mathbf{\Omega}(\mathbf{r}\tau)\cdot\mathbf{\Omega}(\mathbf{r}'\tau)\rangle$ has to be determined self-consistently. To this end, in a way similar to Eq.(18), one should integrate from the effective Hamiltonian in Eq.(17) charge and fermion variables to obtain the action for the spin rotational degrees of freedom:

$$\mathcal{S}[\mathbf{\Omega}] = \frac{J}{4}\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\int_0^\beta d\tau\,[\mathbf{\Omega}(\mathbf{r}\tau)\cdot\mathbf{\Omega}(\mathbf{r}'\tau) - 1] \equiv J\sum_{\langle \mathbf{r}\mathbf{r}'\rangle}\int_0^\beta d\tau\left[\mathbf{S}_\zeta(\mathbf{r}\tau)\cdot\mathbf{S}_\zeta(\mathbf{r}'\tau) - \frac{1}{4}\right], \tag{28}$$

where we made use of the formula $\mathbf{\Omega} = \frac{1}{2}\mathrm{tr}(\mathbf{R}\hat{\sigma}^z\mathbf{R}^\dagger\hat{\boldsymbol{\sigma}})$, while the variables $\mathbf{S}_\zeta(\mathbf{r}\tau) = \frac{1}{2}\sum_{\alpha\gamma}\bar{\zeta}_\alpha(\mathbf{r}\tau)\hat{\boldsymbol{\sigma}}_{\alpha\gamma}\zeta_\gamma(\mathbf{r}\tau)$ are the "bosonic" spins in the complex-projective (CP^1) formulation, see Eq.(11). Here, $J = \frac{4t^2}{U}(n_{h\uparrow} - n_{h\downarrow})^2$ indicates that Coulomb energy U induced Hubbard band splitting is a necessary prerequisite to sustain AF correlations. Although, the fermions $\bar{h}_\alpha(\mathbf{r}\tau), h_\alpha(\mathbf{r}\tau)$ play the role similar to "spinons" in the slave particle formulation a major quantitative difference appears. In SP theory at half-filling "spinons" are paired at $T = 0$ with $\Delta_d \neq 0$. This is clearly impossible here due to the presence Zeeman-type band splitting term in Eq.(17) that marks the onset of charge gap. It prevents a non-zero solution for Δ_d since at half filling ρ is approaching the high energy charge gap, while Δ_d is governed by the exchange energy $J_A \ll U$. We ascribe this discrepancy to the inherent inability in the SP scheme to give an account of the high-energy effects that are the hallmark of the Mott physics.[9] The reason is that the SP theory handels exclusively with with a strictly *low-energy* effective Hamiltonian. This feature is odd e.g. with experiments which show that the superconducting transition in cuprates is accompanied by changes in the optical response, even at energies of the order of 100 times the critical temperature, clearly pointing out the importance of the high-energy effects on the scale given by U.[10] Finally we observe that superconductivity demands more than just paired fermions - it also requires phase coherence in the charge sector distinguished by the variables $z(\mathbf{r}\tau) = e^{i\phi(\mathbf{r}\tau)}$. Therefore a fully self-consistent theory requires a counterpart of the action in Eq.(28) for the phase variables and phase stiffnesses that are responsible for the actual superconducting state.[11]

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